Hilbert spaces of entire functions with trivial zeros

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Abstract

Let H be a Hilbert space of entire functions. Let H' be the space of the functions $f(z)/\prod_i(z-z_i)$ where f belongs to H and vanishes at n given complex points z_i. We compute a suitable E function for H' when one is given for H.

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1 Hilbert space of entire functions with vanishing conditions

Let H be an Hilbert space, whose vectors are entire functions, and such that the evaluations at complex numbers are continuous linear forms, hence correspond to specific vectors Z_z:

$$Z_z \in H, \quad \forall f \in H \qquad (Z_z, f) = f(z) \tag{1}$$

Scalar products (g, f) in this paper will be complex linear in f and conjugate linear in g. Using the Banach-Steinhaus theorem we know that evaluations of derivatives $f \mapsto f^{(k)}(z)$ are also bounded linear forms on the Hilbert space H. We will write $Z_{z,k}$ for the corresponding vectors.

If H satisfies the axiomatic framework of [1] (and is not the zero space), there is an entire function E (not unique) with the property:

$$\mathrm{Im}(z) > 0 \implies |E(z)| > |E(\overline{z})| \tag{2}$$

and in terms of which the evaluators are given as:

$$Z_z(w) = \frac{\overline{E(z)}E(w) - \overline{E^*(z)}E^*(w)}{i(\overline{z} - w)} \tag{3}$$

We have written E^* for the function $w \mapsto \overline{E(\overline{w})}$.

Conversely to each entire function satisfying (2) there is associated a Hilbert space $H(E)$ with evaluators given by (3): the elements of $H(E)$ are the entire functions f such that both f/E and f^*/E belong to the Hardy space of the upper half-plane ($f^*(z) = \overline{f(\overline{z})}$). A basic instance of this theory is the Paley-Wiener space $H = PW_x$ ($x > 0$) of entire functions, square integrable on the real line, and of exponential type at most x. For these classical Paley-Wiener spaces we use the scalar product $(f, f) = \frac{1}{2\pi}\int_{\mathbb{R}} |f(z)|^2\, dz$, and the presence here of a $\frac{1}{2\pi}$ is related to its absence in (3); we find this more convenient. An appropriate E function for PW_x is $E(z) = e^{-ixz}$.

One of the axioms of [1] is: (1) if $f \in H$ verifies $f(z_0) = 0$ then $\frac{z-\overline{z_0}}{z-z_0} f(z)$ belongs to H and has the same norm as f. The others are: (2) H is a Hilbert space and evaluating at a complex point is a bounded linear form, and (3) for any $f \in H$, the function f^* belongs to H and has the same norm as f. From axiom (1) one sees that for z_0 non real, we can always find in H (if it is not the zero space) a function not vanishing at z_0. So the evaluator Z_{z_0} can not be zero if z_0 is non real (and if H is not the zero space). Of course when H is known as an $H(E)$, then $\mathrm{Im}(z) \neq 0 \implies Z_z(z) = \|Z_z\|^2 > 0$ by (3) and (2).

Let $\sigma = (z_1, \dots, z_n)$ be a finite sequence of non necessarily distinct complex numbers with associated evaluators $Z_1, \dots, Z_n$ in H. More precisely, in the case where the z_i's are not all distinct, we assemble in succession the indices corresponding to the same complex number, and if for example $z_1 = z_2 = z_3 \neq z_4$, we let $Z_1 = Z_{z_1}$, $Z_2 = Z_{z_1,1}$, $Z_3 = Z_{z_1,2}$, and $Z_4 = Z_{z_4}$ etc... Also, when f is an arbitrary analytic function, we introduce a notation $f[z_j]$ such that in the example above $f[z_1] = f(z_1)$, $f[z_2] = f'(z_2)(= f'(z_1))$, $f[z_3] = f''(z_3)$, $f[z_4] = f(z_4)$ etc...

Let H^σ be the closed subspace of H of functions vanishing at the z_i's, in other words this is the orthogonal complement to the Z_i's, $1 \leq i \leq n$. For the classical Paley-Wiener spaces $H = PW_x$, such subspaces PW_x^σ have been considered in the work of Lyubarskii and Seip [3], where however the sequences σ arising are infinite. We restrict ourselves here to the finite case which is already interesting: as will be shown in the companion paper [2] this has given a way from the classical Paley-Wiener spaces to explicit Painlevé VI transcendents.

Let

$$\gamma(z) = \frac{1}{(z - z_1)\dots(z - z_n)} \tag{4}$$

and define $H(\sigma) = \gamma(z)H^\sigma$:

$$H(\sigma) = \{F(z) = \gamma(z)f(z) \mid f \in H, f[z_1] = \dots = f[z_n] = 0\} \tag{5}$$

We call $F(z) = \gamma(z)f(z)$ the "complete" form of f (for any f, in H or not, vanishing

at the z_i's), and call $z_1, \ldots, z_n$ the "trivial zeros". We say that we switch from the space H to the space $H(\sigma)$ by adding trivial zeros, but this is of course slightly misleading as the z_i's are trivial zeros only for the incomplete functions $f(z)$, not for the complete functions $F(z)$ which are vectors in the space $H(\sigma)$.

We give $H(\sigma)$ the Hilbert space structure which makes $f \mapsto F$ an isometry with H^σ. Let us note that evaluations $F \mapsto F(z)$ are again continuous linear forms on this new Hilbert space of entire functions: this is immediate if $z \notin \sigma$ and follows from the Banach-Steinhaus theorem if $z \in \sigma$. Let $F \in H(\sigma)$ with incomplete form f. If $F(z_0) = 0$ then $f(z_0) = 0$, with multiplicity suitably increased if z_0 belongs to σ. The function $g(z) = \frac{z-\overline{z_0}}{z-z_0} f(z)$ belongs to H and still vanishes on σ (multiplicities included), hence its complete form $\frac{z-\overline{z_0}}{z-z_0} F(z)$ belongs to $H(\sigma)$. Finally, let us consider for $F \in H(\sigma)$ its conjugate in the real axis $F^*(z) = \overline{F(\overline{z})}$. With $F(z) = \gamma(z) f(z)$ we thus have $F^*(z) = \gamma^*(z) f^*(z) = \gamma(z) \prod_{1 \leq i \leq n} \frac{z-z_i}{z-\overline{z_i}} f^*(z)$. But the function f^* belongs to H (with the same norm as f) and has zeros at the $\overline{z_i}$'s. Hence $\prod_{1 \leq i \leq n} \frac{z-z_i}{z-\overline{z_i}} f^*(z)$ belongs to H with the same norm. And it has zeros at the z_i's, it is thus an element of H^σ and its complete form is an element of $H(\sigma)$ (with the same norm as F). This completes the verification that $H(\sigma)$ verifies the axioms of [1] if H does.

Remark 1. Let us suppose that z_0 is not real. From what precedes if we can find a non-zero element F in $H(\sigma)$ we can find one with $F(z_0) \neq 0$. This proves in particular that if $Z_1, \ldots, Z_n$ do not already span H, then any evaluator Z_{z_0} with z_0 non-real and distinct from the z_i's is not a linear combination of $Z_1, \ldots, Z_n$.

Let us give a first formula (which does not use (3) but only (1)) for the evaluators K_z in $H(\sigma)$ and their scalar products $(K_w, K_z) = K_z(w)$. Let $k_z \in H^\sigma$ be the incomplete form of K_z, so that $K_z(w) = \gamma(w) k_z(w)$. One has to be careful that for $f \in H^\sigma$, with complete form F, we have by definition $(k_z, f) = (K_z, F) = F(z) = \gamma(z) f(z) = \gamma(z)(Z_z, f)$. Hence k_z is $\overline{\gamma(z)}$ times the orthogonal projection $\pi(Z_z)$ of $Z_z \in H$ onto $H^\sigma \subset H$. As is well-known, orthogonal projections can be written in Gram determinantal form:

$$k_z = \overline{\gamma(z)} \pi(Z_z) = \overline{\gamma(z)} \frac{1}{G_n} \begin{vmatrix} (Z_1, Z_1) & \ldots & (Z_1, Z_n) & (Z_1, Z_z) \\ (Z_2, Z_1) & \ldots & (Z_2, Z_n) & (Z_2, Z_z) \\ \vdots & \ldots & \vdots & \vdots \\ Z_1 & \ldots & Z_n & Z_z \end{vmatrix} \tag{6}$$

We wrote G_n for the principal $n \times n$ minor. Then $(K_w, K_z) = K_z(w) = \gamma(w) k_z(w)$, and we have thus obtained, writing now K_z^σ for K_z:

Proposition 1. *Let H be a Hilbert space of entire functions with continuous evaluators Z_z: $\forall f \in H$ $f(z) = (Z_z, f)$. Let $\sigma = (z_1, \ldots, z_n)$ be a finite sequence of (non necessarily distinct) complex numbers with associated evaluators $Z_1, \ldots, Z_n$, assumed to be linearly*

independent. Let $H(\sigma)$ be the Hilbert space of entire functions which are complete forms of the elements of H vanishing on σ. The evaluators of $H(\sigma)$ are given by:

$$K_z^\sigma(w) = \frac{\gamma(w)\overline{\gamma(z)}}{G_n^\sigma} \begin{vmatrix} (Z_1, Z_1) & \dots & (Z_1, Z_n) & (Z_1, Z_z) \\ (Z_2, Z_1) & \dots & (Z_2, Z_n) & (Z_2, Z_z) \\ \vdots & \dots & \vdots & \vdots \\ (Z_w, Z_1) & \dots & (Z_w, Z_n) & (Z_w, Z_z) \end{vmatrix} \tag{7}$$

where G_n^σ is the principal $n \times n$ minor of the matrix at the numerator. Of course this formula must be interpreted as a limit when z or w belongs to σ.

Assume that an E function is known such that the evaluators in H are given by formula (3). We find a function E_σ playing the analogous role for $H(\sigma)$:

Theorem 2. *Let $E_\sigma(w)$ be the unique entire function such that its "incomplete form" (its product with $\prod_{1\leq i\leq n}(w - z_i)$) differs from $E(w)$ by a finite linear combination of the evaluators $Z_1(w)$, ..., $Z_n(w)$. In other words, let*

$$E_\sigma(w) = \frac{\gamma(w)}{G_n^\sigma} \begin{vmatrix} (Z_1, Z_1) & \dots & (Z_1, Z_n) & E[z_1] \\ (Z_2, Z_1) & \dots & (Z_2, Z_n) & E[z_2] \\ \vdots & \dots & \vdots & \vdots \\ Z_1(w) & \dots & Z_n(w) & E(w) \end{vmatrix} \tag{8}$$

where G_n^σ is the principal $n \times n$ minor. The evaluator K_z^σ at z for the space $H(\sigma)$ verifies:

$$K_z^\sigma(w) = (K_w^\sigma, K_z^\sigma) = \frac{\overline{E_\sigma(z)}E_\sigma(w) - \overline{E_\sigma^*(z)}E_\sigma^*(w)}{i(\overline{z} - w)} \tag{9}$$

Remark 2. We mentioned earlier that, if H is not already spanned by the Z_i, $1 \leq i \leq n$, any evaluator Z_z with $\mathrm{Im}(z) \neq 0$ is linearly independent from the Z_i's. This implies $K_z^\sigma \neq 0$, hence for $\mathrm{Im}(z) > 0$ and by (9): $|E_\sigma(z)|^2 - |E_\sigma^*(z)|^2 > 0$, thus E_σ given by (8) indeed verifies (2) if $H(\sigma)$ is not the zero space.

Remark 3. Let us write $F(w) = E^*(w) = \overline{E(\overline{w})}$ and similarly $F_\sigma(w) = E_\sigma^*(w) = \overline{E_\sigma(\overline{w})}$. It will be shown in the proof that F_σ follows the same recipe as E_σ:

$$F_\sigma(w) = \frac{\gamma(w)}{G_n^\sigma} \begin{vmatrix} (Z_1, Z_1) & \dots & (Z_1, Z_n) & F[z_1] \\ (Z_2, Z_1) & \dots & (Z_2, Z_n) & F[z_2] \\ \vdots & \dots & \vdots & \vdots \\ Z_1(w) & \dots & Z_n(w) & F(w) \end{vmatrix} \tag{10}$$

Remark 4. We did not see an immediate easy manipulation of determinants leading to (9) from (7) and (8). Even the compatibility of the two equations (8) and (10) with the relation

$F_\sigma = E_\sigma^*$ does not seem to be immediately visible from easy manipulations of determinants. However, under the additional hypotheses that the space H has the additional symmetry $f(z) \mapsto f(-z)$ and that the "trivial zeros" are purely imaginary and distinct, a relatively simple determinantal approach is proposed in [2]. It leads in fact to other determinantal expressions for E_σ and F_σ than (8) and (10), thus giving further determinantal identities.

2 Adding one zero

We establish the case $n = 1$ of Theorem 2 by direct computation. Appropriate notations are needed in order to complete this deceptively simple looking task. We define:

$$\mathcal{E}(w) = \frac{1}{w - z_1}\left(E(w) - E(z_1)\frac{Z_1(w)}{(Z_1, Z_1)}\right) \tag{11}$$

$$\mathcal{F}(w) = \frac{1}{w - z_1}\left(F(w) - F(z_1)\frac{Z_1(w)}{(Z_1, Z_1)}\right) \tag{12}$$

$$e_1 = E(z_1) \qquad f_1 = F(z_1) \tag{13}$$

$$\text{hence:} \qquad E(w) = (w - z_1)\mathcal{E}(w) + e_1\frac{\overline{e_1}E(w) - \overline{f_1}F(w)}{i(Z_1, Z_1)(\overline{z_1} - w)} \tag{14}$$

$$\Longrightarrow \quad F(w) = (w - \overline{z_1})\mathcal{E}^*(w) - \overline{e_1}\frac{e_1F(w) - f_1E(w)}{i(Z_1, Z_1)(z_1 - w)} \tag{15}$$

$$\text{on the other hand:} \qquad F(w) = (w - z_1)\mathcal{F}(w) + f_1\frac{\overline{e_1}E(w) - \overline{f_1}F(w)}{i(Z_1, Z_1)(\overline{z_1} - w)} \tag{16}$$

We multiply the last identity by $\overline{z_1} - w$, the one before by $z_1 - w$ and substract:

$$(z_1 - \overline{z_1})F(w) = (w - z_1)(\overline{z_1} - w)(\mathcal{E}^*(w) - \mathcal{F}(w)) - \frac{|e_1|^2 - |f_1|^2}{i(Z_1, Z_1)}F(w) \tag{17}$$

Thus $(w - z_1)(\overline{z_1} - w)(\mathcal{E}^*(w) - \mathcal{F}(w)) = 0$ and we have established:

$$\mathcal{F}(w) = \mathcal{E}^*(w) \tag{18}$$

We will also need the following identity:

$$\overline{e_1}\,\mathcal{E}(w) - \overline{f_1}\,\mathcal{F}(w) = -iZ_1(w) \tag{19}$$

Indeed, from (14) and (16):

$$\overline{e_1}E(w) - \overline{f_1}F(w) = (w - z_1)(\overline{e_1}\mathcal{E}(w) - \overline{f_1}\mathcal{F}(w)) + (|e_1|^2 - |f_1|^2)\frac{Z_1(w)}{(Z_1, Z_1)} \tag{20}$$

$$\implies i(\overline{z_1} - w)Z_1(w) = (w - z_1)(\overline{e_1}\mathcal{E}(w) - \overline{f_1}\mathcal{F}(w)) + i(\overline{z_1} - z_1)Z_1(w) \tag{21}$$

This proves (19).

Let us now compute the determinant

$$\begin{vmatrix} (Z_1, Z_1) & (Z_1, Z_z) \\ (Z_w, Z_1) & (Z_w, Z_z) \end{vmatrix} = (Z_1, Z_1)\frac{\overline{E(z)}E(w) - \overline{F(z)}F(w)}{i(\overline{z} - w)} - \overline{Z_1(z)}Z_1(w) \tag{22}$$

We first consider:

$$(Z_1, Z_1)\overline{E(z)}E(w) - (Z_1, Z_1)\overline{F(z)}F(w) - i(\overline{z_1} - w)\overline{Z_1(z)}Z_1(w) \tag{23}$$

$$= (Z_1, Z_1)\overline{E(z)}E(w) - (Z_1, Z_1)\overline{F(z)}F(w) - \overline{Z_1(z)}(\overline{e_1}E(w) - \overline{f_1}F(w)) \tag{24}$$

$$= (Z_1, Z_1)\overline{(z - z_1)}\left(\overline{\mathcal{E}(z)}E(w) - \overline{\mathcal{F}(z)}F(w)\right) \tag{25}$$

Using (14) and (16) this is equal to

$$(Z_1, Z_1)(\overline{z} - \overline{z_1})\left(\overline{\mathcal{E}(z)}(w - z_1)\mathcal{E}(w) + \overline{\mathcal{E}(z)}\frac{e_1 Z_1(w)}{(Z_1, Z_1)} - \overline{\mathcal{F}(z)}(w - z_1)\mathcal{F}(w) - \overline{\mathcal{F}(z)}\frac{f_1 Z_1(w)}{(Z_1, Z_1)}\right) \tag{26}$$

$$= (Z_1, Z_1)(\overline{z} - \overline{z_1})(w - z_1)(\overline{\mathcal{E}(z)}\mathcal{E}(w) - \overline{\mathcal{F}(z)}\mathcal{F}(w)) + (e_1\overline{\mathcal{E}(z)} - f_1\overline{\mathcal{F}(z)})(\overline{z} - \overline{z_1})Z_1(w) \tag{27}$$

$$= (Z_1, Z_1)(\overline{z} - \overline{z_1})(w - z_1)(\overline{\mathcal{E}(z)}\mathcal{E}(w) - \overline{\mathcal{F}(z)}\mathcal{F}(w)) + i(\overline{z} - \overline{z_1})\overline{Z_1(z)}Z_1(w) \tag{28}$$

where (19) was used. Identity of (23) and (28) gives:

$$\begin{aligned} (Z_1, Z_1)\overline{E(z)}E(w) - (Z_1, Z_1)\overline{F(z)}F(w) - i(\overline{z} - w)\overline{Z_1(z)}Z_1(w) \\ = (Z_1, Z_1)(w - z_1)(\overline{z} - \overline{z_1})\left(\overline{\mathcal{E}(z)}\mathcal{E}(w) - \overline{\mathcal{F}(z)}\mathcal{F}(w)\right) \end{aligned} \tag{29}$$

Comparison with (22) gives the final result:

$$\frac{\gamma_1(w)\overline{\gamma_1(z)}}{(Z_1, Z_1)}\begin{vmatrix} (Z_1, Z_1) & (Z_1, Z_z) \\ (Z_w, Z_1) & (Z_w, Z_z) \end{vmatrix} = \frac{\overline{\mathcal{E}(z)}\mathcal{E}(w) - \overline{\mathcal{F}(z)}\mathcal{F}(w)}{i(\overline{z} - w)} \tag{30}$$

As we know from (18) that $\mathcal{F} = \mathcal{E}^*$ this completes the proof of Theorem 2 in the case $n = 1$.

3 General case with distinct trivial zeros

An induction establishes Theorem 2 when the z_i's are distinct. Let us suppose it true for the $n-1$ added "trivial zeros" $z_1, \ldots, z_{n-1}$. From (7) we know that for any $z \in \mathbb{C}\backslash\{z_1, \ldots, z_{n-1}\}$, $\prod_{1\leq i\leq n-1}(w - z_i)K_z^{z_1,\ldots,z_{n-1}}(w)$ is a linear combination of the original evaluators $Z_1(w)$, $\ldots$, $Z_{n-1}(w)$ and $Z_z(w)$. This applies in particular to $z = z_n$. The induction hypothesis

tells us that $\prod_{1\le i<n}(w-z_i)E^{z_1,\dots,z_{n-1}}(w)$ differs from $E(w)$ by a linear combination of the original evaluators $Z_1(w)$, ..., $Z_{n-1}(w)$. The case $n=1$ tells us that $(w-z_n)E^{z_1,\dots,z_n}(w)$ differs from $E^{z_1,\dots,z_{n-1}}(w)$ by a multiple of $K_{z_n}^{z_1,\dots,z_{n-1}}(w)$. Hence $\prod_{1\le i\le n}(w-z_i)E^{z_1,\dots,z_n}(w)$ differs from $E(w)$ by a linear combination of the original evaluators $Z_1(w)$, ..., $Z_{n-1}(w)$ and $Z_n(w)$, as was to be established. This linear combination is fixed in a unique manner by evaluating at the trivial zeros of $\prod_{1\le i\le n}(w-z_i)E^{z_1,\dots,z_n}(w)$. This completes the proof of Theorem 2. Furthermore we proved that the same iterative recipe as for E_σ works for the construction of $F_\sigma = E_\sigma^*$. Hence the formula (10) holds.

4 General case with multiplicities

We introduce some notations for the case of repetitions among $z_1, \dots, z_n$. We define $k_1, \dots, k_n$ such that $i' = i - k_i$ is the first index with $z_{i'} = z_i$. For example if $z_1 = z_2 = z_3 \neq z_4 = z_5$, $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = 0$, $k_5 = 1$, etc... Then, for f an analytic function, we recall the notation $f[z_i] := f^{(k_i)}(z_i)$. For $f \in H$ we also have the scalar product $(Z_i, f) = f[z_i]$.

Let $k^\sigma(z,w)$ be the incomplete form of the reproducing kernel $K^\sigma(z,w)$ in $H(\sigma)$:

$$k^\sigma(z,w) = \frac{1}{G_n}\begin{vmatrix} (Z_1,Z_1) & \dots & (Z_1,Z_n) & (Z_1,Z_z) \\ (Z_2,Z_1) & \dots & (Z_2,Z_n) & (Z_2,Z_z) \\ \vdots & \dots & \vdots & \vdots \\ (Z_w,Z_1) & \dots & (Z_w,Z_n) & (Z_w,Z_z) \end{vmatrix} = Z_z(w) - \sum_{1\le j\le n} \beta_j^\sigma Z_j(w) \tag{31}$$

where the coefficients β_1^σ, ..., β_n^σ (which are also functions of z) are determined by the constraints:

$$\forall i \qquad \sum_{1\le j\le n} \beta_j^\sigma Z_j[z_i] = Z_z[z_i] \quad (= (Z_i, Z_z)) \tag{32}$$

Let $\epsilon > 0$ and $z_i^\epsilon = z_i - k_i\epsilon$ for $1 \le i \le n$. We let $k^\epsilon(z,w)$ be the (completely) incomplete form of the reproducing kernel in $H(z_1^\epsilon, \dots, z_n^\epsilon)$. It thus has the shape:

$$k^\epsilon(z,w) = Z_z(w) - \sum_{1\le j\le n} \alpha_j Z_{z_j^\epsilon}(w) \tag{33}$$

with the constraints

$$\forall i \qquad k^\epsilon(z, z_i - k_i\epsilon) = 0 \tag{34}$$

Let us use the definitions

$$Z_j^\epsilon := \epsilon^{-k_j} \sum_{0\le m\le k_j} (-1)^m \binom{k_j}{m} Z_{z_j - m\epsilon} \tag{35}$$

to obtain vectors $Z_j^\epsilon \in H$ which span the same subspace of H as the $Z_{z_j^\epsilon} = Z_{z_j - k_j\epsilon}$. So there are coefficients β_1^ϵ, ..., β_n^ϵ such that

$$k^\epsilon(z,w) = Z_z(w) - \sum_{1\leq j\leq n} \beta_j^\epsilon Z_j^\epsilon(w) \tag{36}$$

with the constraints

$$\forall i \qquad \sum_{1\leq j\leq n} \beta_j^\epsilon Z_j^\epsilon(z_i - \epsilon k_i) = Z_z(z_i - \epsilon k_i) \tag{37}$$

We define the symbol, for any arbitrary analytic function on $\mathbb{C}$:

$$f[z_i^\epsilon] := \epsilon^{-k_i} \sum_{0\leq l\leq k_i} (-1)^l \binom{k_i}{l} f(z_i - l\epsilon) \tag{38}$$

With the help of these symbols, the constraints on the β_j^ϵ can be equivalently rewritten:

$$\forall i \qquad \sum_{1\leq j\leq n} \beta_j^\epsilon Z_j^\epsilon[z_i^\epsilon] = Z_z[z_i^\epsilon] \tag{39}$$

We examine the behavior for $\epsilon \to 0$ of the quantities $Z_j^\epsilon[z_i^\epsilon]$. In terms of the reproducing kernel $Z(z,w) = Z_z(w) = (Z_w, Z_z)$, which from equation (3) is analytic in w and anti-analytic in z we have:

$$Z_j^\epsilon[z_i^\epsilon] = \frac{1}{\epsilon^{k_i+k_j}} \sum_{\substack{0\leq l\leq k_i \\ 0\leq m\leq k_j}} (-1)^{l+m} \binom{k_i}{l}\binom{k_j}{m} Z(z_j - m\epsilon, z_i - l\epsilon) \tag{40}$$

Thus, with $(\partial F)(w_1,w_2) = \frac{\partial}{\partial w_2}F(w_1,w_2)$, $(\delta F)(w_1,w_2) = \frac{\partial}{\partial \overline{w_1}}F(w_1,w_2)$:

$$\lim_{\epsilon\to 0} Z_j^\epsilon[z_i^\epsilon] = (\partial^{k_i}\delta^{k_j} Z)(z_j, z_i) \tag{41}$$

On the other hand we have:

$$\begin{aligned}(Z_i, Z_j) = \left.\frac{\partial^{k_i}}{\partial w^{k_i}}\right|_{w=z_i} Z_j(w) = \left.\frac{\partial^{k_i}}{\partial w^{k_i}}\right|_{w=z_i} \overline{(Z_j, Z_w)} = \left.\frac{\partial^{k_i}}{\partial w^{k_i}}\right|_{w=z_i} \overline{\left.\frac{\partial^{k_j}}{\partial \omega^{k_j}}\right|_{\omega=z_j} Z(w,\omega)} \\ = \left.\frac{\partial^{k_i}}{\partial w^{k_i}}\right|_{w=z_i} \left.\frac{\partial^{k_j}}{\partial \overline{\omega}^{k_j}}\right|_{\omega=z_j} Z(\omega, w) = (\partial^{k_i}\delta^{k_j} Z)(z_j, z_i)\end{aligned} \tag{42}$$

which gives

$$(Z_i, Z_j) = (\partial^{k_i}\delta^{k_j} Z)(z_j, z_i) = \lim_{\epsilon\to 0} Z_j^\epsilon[z_i^\epsilon] \tag{43}$$

There also holds, for any z:

$$(Z_i, Z_z) = \lim_{\epsilon\to 0} Z_z[z_i^\epsilon] \tag{44}$$

So in the limit $\epsilon \to 0$ the linear constraints (39) on $(\beta_j^\epsilon)_{1\le j\le n}$ become the constraints on the coefficients β_1^σ, ..., β_n^σ which give in (31) the incomplete reproducing kernel $k^\sigma(z,w)$. This shows in passing that for $\epsilon \neq 0$ small the vectors Z_j^ϵ are also linearly independent, and proves $\forall j\ \lim_{\epsilon\to 0}\beta_j^\epsilon = \beta_j^\sigma$. We have further

$$Z_j(w) = \lim_{\epsilon\to 0} Z_j^\epsilon(w) \tag{45}$$

and this finally establishes:

$$\lim_{\epsilon\to 0} k^\epsilon(z,w) = k^\sigma(z,w) \tag{46}$$

In the exact same manner we can examine the quantities:

$$e_\sigma(w) = \frac{1}{G_n^\sigma}\begin{vmatrix} (Z_1,Z_1) & \dots & (Z_1,Z_n) & E[z_1] \\ (Z_2,Z_1) & \dots & (Z_2,Z_n) & E[z_2] \\ \vdots & \dots & \vdots & \vdots \\ Z_1(w) & \dots & Z_n(w) & E(w) \end{vmatrix} \tag{47}$$

and

$$e_\epsilon(w) = \frac{1}{G_n^\epsilon}\begin{vmatrix} (Z_{z_1^\epsilon},Z_{z_1^\epsilon}) & \dots & (Z_{z_1^\epsilon},Z_{z_n^\epsilon}) & E(z_1^\epsilon) \\ (Z_{z_2^\epsilon},Z_{z_1^\epsilon}) & \dots & (Z_{z_2^\epsilon},Z_{z_n^\epsilon}) & E(z_2^\epsilon) \\ \vdots & \dots & \vdots & \vdots \\ Z_{z_1^\epsilon}(w) & \dots & Z_{z_n^\epsilon}(w) & E(w) \end{vmatrix} \tag{48}$$

and prove

$$\lim_{\epsilon\to 0} e_\epsilon(w) = e_\sigma(w) \tag{49}$$

There is also an immediate limit to be taken in the gamma factor, and in the end we obtain the reproducing kernel formula (9) for the space $H(\sigma)$. The formula $E_\sigma^* = F_\sigma$ with F_σ given by (10) is proven in the same manner.

5 An example

We take $H = PW_x$ $(x > 0)$, the Paley-Wiener space of entire functions of exponential type at most x, square integrable on the real line $((f,f) = \frac{1}{2\pi}\int_{\mathbb{R}} |f(z)|^2\,dz)$. The evaluators are given by the formula

$$Z_z(w) = (Z_w, Z_z) = 2\frac{\sin((\overline{z}-w)x)}{\overline{z}-w} \quad (= \int_{-x}^{x} e^{iwt}e^{-i\overline{z}t}\,dt) \tag{50}$$

We can choose $E(z) = e^{-ixz}$, $F(z) = E^*(z) = e^{+ixz}$. Let z_1, z_2, ..., z_n and z be distinct complex numbers and define

$$G_n = \begin{vmatrix} 2\frac{\sin((\overline{z_1}-z_1)x)}{\overline{z_1}-z_1} & 2\frac{\sin((\overline{z_2}-z_1)x)}{\overline{z_2}-z_1} & \dots & 2\frac{\sin((\overline{z_n}-z_1)x)}{\overline{z_n}-z_1} \\ 2\frac{\sin((\overline{z_1}-z_2)x)}{\overline{z_1}-z_2} & 2\frac{\sin((\overline{z_2}-z_2)x)}{\overline{z_2}-z_2} & \dots & 2\frac{\sin((\overline{z_n}-z_2)x)}{\overline{z_n}-z_2} \\ \vdots & \dots & \dots & \vdots \\ 2\frac{\sin((\overline{z_1}-z_n)x)}{\overline{z_1}-z_n} & \dots & \dots & 2\frac{\sin((\overline{z_n}-z_n)x)}{\overline{z_n}-z_n} \end{vmatrix} \tag{51}$$

$$G_n(z,z) = \begin{vmatrix} 2\frac{\sin((\overline{z_1}-z_1)x)}{\overline{z_1}-z_1} & 2\frac{\sin((\overline{z_2}-z_1)x)}{\overline{z_2}-z_1} & \dots & 2\frac{\sin((\overline{z_n}-z_1)x)}{\overline{z_n}-z_1} & 2\frac{\sin((\overline{z}-z_1)x)}{\overline{z}-z_1} \\ 2\frac{\sin((\overline{z_1}-z_2)x)}{\overline{z_1}-z_2} & 2\frac{\sin((\overline{z_2}-z_2)x)}{\overline{z_2}-z_2} & \dots & 2\frac{\sin((\overline{z_n}-z_2)x)}{\overline{z_n}-z_2} & 2\frac{\sin((\overline{z}-z_2)x)}{\overline{z}-z_2} \\ \vdots & \dots & \vdots & \vdots & \vdots \\ 2\frac{\sin((\overline{z_1}-z)x)}{\overline{z_1}-z} & \dots & \dots & 2\frac{\sin((\overline{z_n}-z)x)}{\overline{z_n}-z} & 2\frac{\sin((\overline{z}-z)x)}{\overline{z}-z} \end{vmatrix} \tag{52}$$

$$e_n(z) = \begin{vmatrix} 2\frac{\sin((\overline{z_1}-z_1)x)}{\overline{z_1}-z_1} & 2\frac{\sin((\overline{z_2}-z_1)x)}{\overline{z_2}-z_1} & \dots & 2\frac{\sin((\overline{z_n}-z_1)x)}{\overline{z_n}-z_1} & e^{-ixz_1} \\ 2\frac{\sin((\overline{z_1}-z_2)x)}{\overline{z_1}-z_2} & 2\frac{\sin((\overline{z_2}-z_2)x)}{\overline{z_2}-z_2} & \dots & 2\frac{\sin((\overline{z_n}-z_2)x)}{\overline{z_n}-z_2} & e^{-ixz_2} \\ \vdots & \dots & \vdots & \vdots & \vdots \\ 2\frac{\sin((\overline{z_1}-z)x)}{\overline{z_1}-z} & \dots & \dots & 2\frac{\sin((\overline{z_n}-z)x)}{\overline{z_n}-z} & e^{-ixz} \end{vmatrix} \tag{53}$$

$$f_n(z) = \begin{vmatrix} 2\frac{\sin((\overline{z_1}-z_1)x)}{\overline{z_1}-z_1} & 2\frac{\sin((\overline{z_2}-z_1)x)}{\overline{z_2}-z_1} & \dots & 2\frac{\sin((\overline{z_n}-z_1)x)}{\overline{z_n}-z_1} & e^{ixz_1} \\ 2\frac{\sin((\overline{z_1}-z_2)x)}{\overline{z_1}-z_2} & 2\frac{\sin((\overline{z_2}-z_2)x)}{\overline{z_2}-z_2} & \dots & 2\frac{\sin((\overline{z_n}-z_2)x)}{\overline{z_n}-z_2} & e^{ixz_2} \\ \vdots & \dots & \vdots & \vdots & \vdots \\ 2\frac{\sin((\overline{z_1}-z)x)}{\overline{z_1}-z} & \dots & \dots & 2\frac{\sin((\overline{z_n}-z)x)}{\overline{z_n}-z} & e^{ixz} \end{vmatrix} \tag{54}$$

Then

$$G_n(z,z)G_n = \frac{|e_n(z)|^2 - |f_n(z)|^2}{2\operatorname{Im}(z)} \qquad \text{and} \qquad f_n(z) = \prod_{1\le i\le n} \frac{z - z_i}{z - \overline{z_i}}\, \overline{e_n(\overline{z})} \tag{55}$$

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